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On a Paper of Mazhar and Totik*

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For modified Szász operators, S. M. Mazhar and V. Totik gave a direct approximation theorem for continuous functions. In this paper we extend this direct result to combinations of these operators. An inverse theorem to this direct estimate is given. An equivalent relation between the derivatives of these operators and smoothness of functions is also presented.

1. INTRODUCTION

The well-known Bernstein polynomials are given by

$$B_n(f, x) = \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k} \equiv \sum_{k=0}^{n} f(k/n) b_{n,k}(x). \tag{1.1}$$

It was shown by H. Berens and G. G. Lorentz [2] that if $0 < \alpha < 2$ then $|B_n(f, x) - f(x)| \leqslant M(x(1-x)/n)^{\alpha/2}$ if and only if $\|f(x+h) - 2f(x) + f(x-h)\|_{C[h, 1-h]} = O(h^\alpha)$. M. Becker [1] extended this result to certain other exponential type operators, which also gave the characterization of the classical modulus of smoothness. It is thus of interest to characterize the higher orders of smoothness by positive linear operators. In this paper we give such a characterization by the so-called modified Szász operators on $[0, \infty)$

$$\begin{aligned} L_n(f, x) &= \sum_{k=0}^{\infty} n \int_0^\infty f(t) P_{n,k}(t)\, dt\, P_{n,k}(x), \\ P_{n,k}(x) &= e^{-nx} (nx)^k / k!. \end{aligned} \tag{1.2}$$

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These operators were defined by S. M. Mazhar and V. Totik [9] in 1985. The saturation and non-optimal characterization results were given by means of the so-called Ditzian–Totik modulus of smoothness.

Mazhar and Totik gave a direct theorem in the form

$$|L_n(f, x) - f(x)| \leqslant M\omega_1(f, (x/n + n^{-2})^{1/2}). \tag{1.3}$$

The inverse result to (1.3) was considered as "far less obvious." We shall give such an inverse result. We shall also extend (1.3) and the inverse theorem to combinations of these operators as defined by M. Heilmann [8] in 1989, which have higher orders of approximation.

Another interesting topic in this area is the close connection between smoothness of functions and the behavior of derivatives of optimal polynomials or approximation processes [5, 7]. In this paper we give a similar result for the modified Szász operators.

2. Notations and Preliminary Results

Let $C[0, \infty)$ be the set of continuous and bounded functions on $[0, \infty)$. $\|\cdot\|$ denotes the supremum norm. The rth modulus of smoothness we will consider is defined in $C[0, \infty)$ by

$$\omega_r(f, t) = \sup_{0 < h \leqslant t} \|\Delta_h^r f(\cdot)\|, \tag{2.1}$$

where $\Delta_h^r f(x) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(x + (k - r/2)h)$, if $x > rh/2$; $\Delta_h^r f(x) = 0$, otherwise.

To characterize smoothness we use a combination of the modified Szász operators given by

$$L_{n,r}(f, x) = \sum_{i=0}^{r-1} a_i(n) L_{n_i}(f, x), \tag{2.2}$$

where with an absolute constant A, n_i, and $a_i(n)$ satisfy

$$\begin{aligned} &\text{(a)} \quad n = n_0 < \cdots < n_{r-1} \leqslant An; \qquad \text{(b)} \quad \sum_{i=0}^{r-1} |a_i(n)| \leqslant A; \\ &\text{(c)} \quad \sum_{i=0}^{r-1} a_i(n) = 1; \qquad \text{(d)} \quad \sum_{i=0}^{r-1} a_i(n) n_i^{-k} = 0, \qquad \text{for} \quad k = 1, ..., r-1. \end{aligned} \tag{2.3}$$

Note that $L_{n,1}(f, x) = L_n(f, x)$.

We also use the Peetre's K-functional

$$K_r(f, t^r) = \inf_{g \in D_r} \{ \|f-g\| + t^r \|g\|_{D_r} \}, \tag{2.4}$$

where the Sobolev space D_r and its norm are defined by

$$D_r = \{ g \in C[0, \infty) : g^{(r-1)} \in \mathrm{AC}_{\mathrm{loc}},\ g^{(r)} \in L_\infty[0, \infty) \},$$
$$\|g\|_{D_r} = \|g\| + \|g^{(r)}\|_\infty.$$

It is well known that for $f \in C[0, \infty)$ we have

$$M_0^{-1} \omega_r(f, t) \leqslant K_r(f, t^r) \leqslant M_0 \omega_r(f, t) \tag{2.5}$$

with a constant M_0 independent of f and $t > 0$.

3. A Direct Theorem

We now extend the direct estimate (1.3) to the combinations $L_{n,r}$. We use the following lemma.

Lemma 3.1 [8]. *Let $W_{n,m}(x) = L_n((x - \cdot)^m, x)$ for $n \in N$, $m \geqslant 0$. Then we have*

$$W_{n,2m}(x) = \sum_{i=0}^{m} q_{i,2m} (x/n)^{m-i} n^{-2i}, \tag{3.1}$$

$$W_{n,2m+1}(x) = \sum_{i=1}^{m} q_{i,2m+1} (x/n)^{m-i} n^{-2i-1}, \tag{3.2}$$

where the real coefficients $\{q_{i,2m}\}$ and $\{q_{i,2m+1}\}$ are independent of x and bounded uniformly by M_1.

Theorem 1. *Let $f \in C[0, \infty)$, $r \in N$. Then we have*

$$\begin{aligned} |L_{n,r}(f, x) - f(x)| &\leqslant M K_r(f, (x/n + n^{-2})^{r/2}) \\ &\leqslant M' \omega_r(f, \sqrt{x/n + n^{-2}}), \end{aligned} \tag{3.3}$$

where M and M' are constants independent of f, $n \in N$, and $x \geqslant 0$.

Proof. From the definition of the $L_{n,r}$ and Lemma 3.1, we have

$$L_{n,r}((\cdot - x)^k, x) = 0, \qquad k = 1, ..., r-1.$$

Let $g \in D_r$. Then by (3.1) and Hölder's inequality we have

$$
\begin{aligned}
|L_{n,r}(g,x)-g(x)| &= \left| L_{n,r}\left(\int_x^t (t-u)^{r-1} g^{(r)}(u)\,du/(r-1)!, x \right) \right| \\
&\leqslant \sum_{i=0}^{r-1} |a_i(n)|\, L_{n_i}(|t-x|^r, x)\, \|g^{(r)}\|_\infty \\
&\leqslant \sum_{i=0}^{r-1} |a_i(n)|\, (L_{n_i}((t-x)^{2r}, x))^{1/2}\, \|g^{(r)}\|_\infty \\
&\leqslant \sum_{i=0}^{r-1} |a_i(n)|\, (M_1(r+1)(x/n+n^{-2})^r)^{1/2}\, \|g^{(r)}\|_\infty \\
&\leqslant A(M_1(r+1))^{1/2}\,(x/n+n^{-2})^{r/2}\, \|g^{(r)}\|_\infty .
\end{aligned}
$$

Thus, for $f \in C[0,\infty)$, $g \in D_r$, $x \in [0,\infty)$ we have

$$
\begin{aligned}
&|L_{n,r}(f,x)-f(x)| \\
&\quad\leqslant |L_{n,r}(f-g,x)| + |f(x)-g(x)| + |L_{n,r}(g,x)-g(x)| \\
&\quad\leqslant (A+1)\,\|f-g\| + A(M_1(r+1))^{1/2}\,(x/n+n^{-2})^{r/2}\, \|g^{(r)}\|_\infty \\
&\quad\leqslant M\{\|f-g\| + (x/n+n^{-2})^{r/2}\, \|g^{(r)}\|_\infty\}.
\end{aligned}
$$

By taking the infimum over $g \in D_r$, we obtain

$$|L_{n,r}(f,x)-f(x)| \leqslant MK_r(f,(x/n+n^{-2})^{r/2}),$$

where M is a constant independent of f, $x \in [0,\infty)$ and $n \in N$. Using (2.5) we obtain (3.3). Our proof is then complete.

Remark. In the case $r=1$, our result is the direct estimate of Mazhar and Totik [9].

4. An Inverse Theorem

The purpose of this section is to give an inverse result. Some of the ideas used may be found in [1, 7, 12].

THEOREM 2. *Let $f \in C[0,\infty)$, $r \in N$, $0<\alpha<r$. Then we have*

$$|L_{n,r}(f,x)-f(x)| \leqslant C(x/n+n^{-2})^{\alpha/2} \tag{4.1}$$

with a constant C independent of x and n, if and only if

$$\omega_r(f,h) = O(h^\alpha).$$

Remark. In [12] we have shown that for the Bernstein–Durrmeyer operators

$$D_n(f, x) = \sum_{k=0}^{n} (n+1) \int_0^1 f(t) \binom{n}{k} t^k (1-t)^{n-k} \, dt \binom{n}{k} x^k (1-x)^{n-k}, \qquad (4.2)$$

and $0 < \alpha < 1$, we have

$$\omega_1(f, h) = O(h^\alpha) \Leftrightarrow |D_n(f, x) - f(x)| \leqslant C(x(1-x)/n + n^{-2})^{\alpha/2}. \qquad (4.3)$$

The term $x(1-x)/n + n^{-2}$ cannot be replaced by $x(1-x)/n$ in (4.3). Therefore, in (4.1) the term $x/n + n^{-2}$ cannot be replaced by x/n.

We have also stated in [12] that for $1 < \alpha < 2$ there exist no functions $\{\Psi_{n,\alpha}(x)\}_{n \in N}$ such that for $f \in C[0, 1]$ the following equivalence holds

$$\omega_2(f, h) = O(h^\alpha) \Leftrightarrow |D_n(f, x) - f(x)| \leqslant C\Psi_{n,\alpha}(x). \qquad (4.4)$$

In view of this fact we cannot expect a similar characterization by the modified Szász operators for the functions satisfying $\omega_2(f, h) = O(h^\alpha)$ with $1 < \alpha < 2$.

Proof of Theorem 2. It is sufficient to prove the inverse part.

Suppose that (4.1) holds. Let $0 < t \leqslant h \leqslant 1/(8r)$, $n \in N$, $x \in (rt/2, \infty)$. Set $d(n, x, t) = \max\{1/n, \sqrt{(x + rt/2)/n}\}$. Then we have

$$\begin{aligned}
|\Delta_t^r f(x)| &\leqslant \sum_{j=0}^{r} \binom{r}{j} |L_{n,r}(f, x + (j - r/2)t) - f(x + (j - r/2)t)| \\
&\quad + \left| \int \cdots \int_{-t/2}^{t/2} L_{n,r}^{(r)}\left(f, x + \sum_{j=1}^{r} u_j\right) du_1 \cdots du_r \right| \\
&\leqslant 2^{2r} C(d(n, x, t))^\alpha \\
&\quad + \sum_{i=0}^{r-1} |a_i(n)| \int \cdots \int_{-t/2}^{t/2} \left\{ \left| L_{n_i}^{(r)}\left(f - f_d, x + \sum_{j=1}^{r} u_j\right) \right| \right. \\
&\quad \left. + \left| L_{n_i}^{(r)}\left(f_d, x + \sum_{j=1}^{r} u_j\right) \right| \right\} du_1 \cdots du_r. \qquad (4.5)
\end{aligned}$$

Here $f_d \in D_r$ is taken, for $d > 0$, to satisfy

$$\|f - f_d\|_\infty \leqslant 2K_r(f, d^r) \leqslant 2M_0 \omega_r(f, d) \qquad (4.6)$$

and

$$\|f_d^{(r)}\|_\infty \leqslant 2d^{-r} K_r(f, d^r) \leqslant 2M_0 d^{-r} \omega_r(f, d). \qquad (4.7)$$

Note that from [8]

$$L_n^{(r)}(f_d, x) = \sum_{k=0}^{\infty} n \int_0^{\infty} P_{n,k+r}(t) f_d^{(r)}(t)\, dt\, P_{n,k}(x).$$

We thus have

$$\begin{aligned}\sum_{i=0}^{r-1} |a_i(n)| \int \cdots \int_{-t/2}^{t/2} \left| L_{n_i}^{(r)}\left(f_d, x + \sum_{j=1}^{r} u_j\right)\right| du_1 \cdots du_r \\ \leqslant At^r \, \|f_d^{(r)}\|_{\infty} \\ \leqslant 2M_0 At^r d^{-r} \omega_r(f, d).\end{aligned} \tag{4.8}$$

By [8] we also have

$$\begin{aligned}|L_{n_i}^{(r)}(f - f_d, x)| &= \left| n_i^r \sum_{k=0}^{\infty} P_{n_i,k}(x) \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} n_i \int_0^{\infty} P_{n_i,k+j}(t) \right. \\ &\qquad \left. \times (f(t) - f_d(t))\, dt \right| \\ &\leqslant 2^r n_i^r \, \|f - f_d\|.\end{aligned} \tag{4.9}$$

Combining this estimate with the result of Lemma 4.1 we obtain

$$\begin{aligned}&\sum_{i=0}^{r-1} |a_i(n)| \int \cdots \int_{-t/2}^{t/2} \left| L_{n_i}^{(r)}\left(f - f_d, x + \sum_{j=1}^{r} u_j\right)\right| du_1 \cdots du_r \\ &\quad \leqslant A(2^r A^r + M_2 A^r) \, \|f - f_d\| \\ &\qquad \times \min\left\{ n^r t^r, n^{r/2} \int \cdots \int_{-t/2}^{t/2} \left(x + \sum_{j=1}^{r} u_j\right)^{-r/2} du_1 \cdots du_r \right\} \\ &\quad \leqslant A^{r+1}(2^r + M_2)(1 + M'_r) \, \|f - f_d\| \, t^r \min\{n^r, (n/(x + rt/2))^{r/2}\} \\ &\quad = A^{r+1}(2^r + M_2)(1 + M'_r) \, \|f - f_d\| \, t^r (d(n, x, t))^{-r}.\end{aligned} \tag{4.10}$$

Here we have used the following inequality which can be proved in the same way as in [12]

$$\int \cdots \int_{-t/2}^{t/2} \left(x + \sum_{j=1}^{r} u_j\right)^{-r/2} du_1 \cdots du_r \leqslant M'_r t^r (x + rt/2)^{-r/2}. \tag{4.11}$$

Thus, from (4.5), (4.8), and (4.10) we obtain

$$\begin{aligned}|\Delta_t^r f(x)| &\leqslant 4^r C(d(n, x, t))^{\alpha} + 2M_0 At^r d^{-r} \omega_r(f, d) \\ &\qquad + 2M_0 A^{r+1}(2^r + M_2)(1 + M'_r) t^r (d(n, x, t))^{-r} \omega_r(f, d).\end{aligned}$$

Let $d=d(n, x, t)$. Then we have

$$|\Delta_t^r f(x)| \leqslant C'\{(d(n, x, t))^\alpha + t^r(d(n, x, t))^{-r}\omega_r(f, d(n, x, t))\},$$

where the constant C' is independent of x, t, h, and n.
Note that for $n \geqslant 2$

$$d(n, x, t) < d(n-1, x, t) \leqslant 2d(n, x, t).$$

Thus for any $\delta \in (0, 1/(8r))$ we can choose $n \in N$ such that

$$d(n, x, t) \leqslant \delta < d(n-1, x, t) \leqslant 2d(n, x, t).$$

Therefore, we have

$$|\Delta_t^r f(x)| \leqslant 2^r C'\{\delta^\alpha + h^r \delta^{-r}\omega_r(f, \delta)\}.$$

Hence

$$\omega_r(f, h) \leqslant 2^r C'\{\delta^\alpha + h^r \delta^{-r}\omega_r(f, d)\},$$

which implies [2, 7]

$$\omega_r(f, h) = O(h^\alpha).$$

Our proof will be complete after we prove Lemma 4.1.

LEMMA 4.1. *If $f \in C[0, \infty)$ and $r \in N$, then*

$$\|x^{r/2} L_n^{(r)}(f, x)\|_\infty \leqslant M_2 n^{r/2} \|f\|, \tag{4.12}$$

where M_2 is a constant independent of f and $n \in N$.

Proof. If r is even, then (4.12) is valid by [8].
Suppose that $r = 2m-1$ with $m \in N$. We have by [7, 8]

$$L_n^{(2m-2)}(f, x) = x^{2-2m} \sum_{i=0}^{2m-2} Q_i(nx) n^i$$
$$\times \sum_{k=0}^{\infty} P_{n,k}(x)(k/n - x)^i n \int_0^\infty f(t) P_{n,k}(t)\, dt,$$

where $Q_i(nx)$ is a polynomial in nx of degree $[(2m-2-i)/2]$ with uniformly bounded constant coefficients. By differentiating the four parts, respectively, and using the moments of Szász–Mirakjan operators in [7] we can obtain the estimate (4.12) for $x \in [1/n, \infty)$. For $x \in (0, 1/n)$ we have

$$|x^{r/2} L_n^{(r)}(f, x)| \leqslant n^{-r/2} \|L_n^{(r)}(f)\|_\infty \leqslant 2^r n^{r/2} \|f\|.$$

The proof is then complete.

5. A Connection between Derivatives and Smoothness

An equivalent relation between the derivatives of the modified Szász operators and the Ditzian–Totik modulus of smoothness was given by M. Heilmann [8], Z. Ditzian, and K. G. Ivanov [6]. In this section we give an equivalent relation between the derivatives and the classical modulus of smoothness. The commutativity of these operators is crucial in our proof.

To prove our main result we need the following lemma.

Lemma 5.1. *Let $r \in N$, $0<s<r$. If $f \in D_r$, then*

$$\|x^{(r-s)/2} L_n^{(r)}(f, x)\|_\infty \leqslant \|x^{(r-s)/2} f^{(r)}(x)\|_\infty. \tag{5.1}$$

Proof. Note that [8]

$$L_n^{(r)}(f, x) = \sum_{k=0}^{\infty} P_{n,k}(x) n \int_0^\infty P_{n,k+r}(t) f^{(r)}(t)\, dt. \tag{5.2}$$

By Hölder's inequality we have

$$\begin{aligned}
&|x^{(r-s)/2} L_n^{(r)}(f, x)| \\
&\leqslant \sum_{k=0}^{\infty} P_{n,k}(x) x^{(r-s)/2} n \int_0^\infty P_{n,k+r}(t) t^{(s-r)/2}\, dt \\
&\quad \times \|x^{(r-s)/2} f^{(r)}(x)\|_\infty \\
&\leqslant \|x^{(r-s)/2} f^{(r)}(x)\|_\infty \left(\sum_{k=0}^{\infty} P_{n,k}(x) x^r n \int_0^\infty P_{n,k+r}(t) t^{-r}\, dt \right)^{(r-s)/(2r)} \\
&\leqslant \|x^{(r-s)/2} f^{(r)}(x)\|_\infty.
\end{aligned}$$

Hence (5.1) holds.

We can now state the main result of this section.

Theorem 3. *Let $f \in C[0, \infty)$, $r \in N$, $0<\alpha<r$. We have*

$$|L_n^{(r)}(f, x)| \leqslant M(\min\{n^2, n/x\})^{(r-\alpha)/2} \Leftrightarrow \omega_r(f, h) = O(h^\alpha). \tag{5.3}$$

Proof. Sufficiency. Let $g \in D_r$, $x \in [0, \infty)$, $n \in N$. By (4.9), (4.12), and (5.2) we have

$$\begin{aligned}
|L_n^{(r)}(f, x)| &\leqslant |L_n^{(r)}(f-g, x)| + |L_n^{(r)}(g, x)| \\
&\leqslant \min\{2^r n^r, M_2 (n/x)^{r/2}\} \|f-g\| + \|g^{(r)}\|_\infty \\
&\leqslant (2^r + M_2)(\min\{n^2, n/x\})^{r/2} \\
&\quad \times \{\|f-g\| + (\min\{n^2, n/x\})^{-r/2} \|g^{(r)}\|_\infty\}.
\end{aligned}$$

Therefore, by taking the infimum over $g \in D_r$ we obtain

$$|L_n^{(r)}(f, x)| \leqslant M_0(2^r + M_2)(\min\{n^2, n/x\})^{r/2} \omega_r(f, (\min\{n^2, n/x\})^{-1/2}).$$

The proof of the sufficiency is complete.

Necessity. Let $0 < t \leqslant h < 1/(8r)$, $x > rt/2$. Set $d(n, x, t) = \max\{1/n, \sqrt{(x + rt/2)/n}\}$. By [8] we have the commutative property

$$L_n(L_m f)(x) = L_m(L_n f)(x), \qquad \text{for} \quad m, n \in N. \tag{5.4}$$

By Theorem 1 we then have

$$\begin{aligned}
|\Delta_t^r L_m f(x)| \leqslant & \left| \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} \{L_{n,r}(L_m f, x + (j - r/2)t) \right. \\
& \left. - L_m(f, x + (j - r/2)t)\} \right| \\
& + \left| \sum_{i=0}^{r-1} a_i(n) \Delta_t^r L_{n_i}(L_m f)(x) \right| \\
\leqslant & \sum_{j=0}^{r} \binom{r}{j} M' \omega_r(L_m f, \sqrt{(x + (j - r/2)t)/n + n^{-2}}) \\
& + \sum_{i=0}^{r-1} |a_i(n)| \int \cdots \int_{-t/2}^{t/2} \left| L_m^{(r)} \left(L_{n_i} f, x + \sum_{j=1}^{r} u_j \right) \right| du_1 \cdots du_r.
\end{aligned}$$

By (5.1) and (5.2) we obtain

$$\begin{aligned}
|\Delta_t^r L_m f(x)| \leqslant & \, 4^r M' \omega_r(L_m f, d(n, x, t)) + \sum_{i=0}^{r-1} |a_i(n)| \\
& \times \min \left\{ \|L_{n_i}^{(r)}(f)\|_\infty t^r, \int \cdots \int_{-t/2}^{t/2} \left(x + \sum_{j=1}^{r} u_j \right)^{(\alpha - r)/2} \right. \\
& \left. \times \|x^{(r - \alpha)/2} L_{n_i}^{(r)}(f, x)\|_\infty du_1 \cdots du_r \right\} \\
\leqslant & \, 4^r M' \omega_r(L_m f, d(n, x, t)) + \sum_{i=0}^{r-1} |a_i(n)| \min\{M(An)^{r-\alpha} t^r, \\
& M(An)^{(r-\alpha)/2} (M_r')^{(r-\alpha)/r} (x + rt/2)^{(\alpha - r)/2} t^r\} \\
\leqslant & \, M''\{\omega_r(L_m f, d(n, x, t)) + t^r (d(n, x, t))^{\alpha - r}\}.
\end{aligned} \tag{5.5}$$

Here we have used (4.11). The constant M'' is independent of m, n, x, and t. For any $\delta \in (0, 1/(8r))$ we can choose $n \in N$ such that

$$d(n, x, t) \leqslant \delta < 2d(n, x, t).$$

Hence

$$|\Delta_t^r L_m f(x)| \leqslant 2^r M'' \{\omega_r(L_m f, \delta) + t^r \delta^{\alpha - r}\},$$

which implies

$$\omega_r(L_m f, h) \leqslant 2^r M'' (\omega_r(L_m f, \delta) + h^r \delta^{\alpha - r}). \tag{5.6}$$

Let $T = (2^{r+1} M'' + 1)^{1/\alpha}$, $\delta = h/T$. Then by induction we obtain for $k \in N$

$$\begin{aligned}
\omega_r(L_m f, h) &\leqslant 2^r M'' (\omega_r(L_m f, h/T) + h^\alpha T^{r-\alpha}) \\
&\leqslant \cdots \\
&\leqslant (2^r M'')^k \omega_r(L_m f, hT^{-k}) + h^\alpha T^r \sum_{l=1}^{k} (2^r M'' T^{-\alpha})^l \\
&\leqslant h^r (2^r M'' T^{-r})^k \, \|L_m^{(r)}(f)\|_\infty + h^\alpha T^r 2^r M''/(T^\alpha - 2^r M'').
\end{aligned}$$

Letting $k \to \infty$ we have

$$\omega_r(L_m f, h) \leqslant T^r h^\alpha,$$

where the constant T^r is independent of $m \in N$ and $h \in (0, 1/(8r))$. Thus we obtain for $x > rt/2$

$$\begin{aligned}
|\Delta_t^r f(x)| &= \lim_{m \to \infty} |\Delta_t^r L_m f(x)| \\
&\leqslant T^r t^\alpha,
\end{aligned}$$

which implies

$$\omega_r(f, h) \leqslant T^r h^\alpha.$$

The proof is complete.

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